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SOLUTIONS OF EQUATIONS FOR THE THERMAL BOUNDARY LAYER AT A ROTATING AXISYMMETRIC SURFACE

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РЕШЕНИЕ УРАВНЕНИЙ ТЕПЛОВОГО ПОГРАНИЧНОГО СЛОЯ, ОБРАЗУЮЩЕГОСЯ НА ВРАЩАЮЩЕЙСЯ ОСЕСИММЕТРИЧНОЙ ПОВЕРХНОСТИ

Аннотация— Решение задачи для сжимаемого газа разыскивается в случае линейной зависимости вязкости от температуры путем применения преобразования Дородницына. Распределения температуры и составляющих вектора скорости представляются в виде рядов относительно параметров, описывающих форму меридионального контура поверхности. В результате этого получается реккурентная система обыкновенных дифференциальных уравнений для коэффициентов при этих параметрах, являющихся функциями безразмерного расстояния от поверхности. Приводятся результаты численного решения на электронной вычислительной машине краевой задачи для указанной системы дифференциальных уравнений. Дается сопоставление данных, основанных на полученном решении, для сферической поверхности с результатами других решений.

NOMENCLATURE

x, distance along the generating line;
y, transversal direction;
Z, distance along the normal to the surface;
r, distance from the rotation axis;
β, half of the expansion angle of the conical nose;
α $= \sin\beta$
u, v, w, velocity vector components in the direction x, y, z;
T, temperature;
$i = c_pT$, enthalpy;
$\varkappa$, isentropic index;
ρ, density;
ν, kinematic viscosity, $\nu = \mu/\rho$;
λ, thermal conductivity;
ω, angular velocity;
ψ, stream function;
x^0, z^0, ζ, defined by equations (5) and (12);
$\varepsilon_K(x)$, shape parameters by equation (10);
$f_i(\zeta), g_i(\zeta), t_i(\zeta), s_i(\zeta)$, coefficients in series expansion from (11);
τ_x, τ_y, components of tangential stress;
θ, $= r^2\omega^2:(i_w - i_\infty)$;
$\bar{\tau}_x$, $= \tau_x:\rho_0(r\omega)^2$, $\bar{\tau}_y = \tau_y:\rho_0(r\omega)^2$;
c_m, $= \dfrac{4\pi \int_0^{x_m} r^2\tau_y\,dx}{\rho_\infty r_m^4 \nu_\infty^{\frac{1}{2}} \omega^{\frac{3}{2}}}$, drag coefficient;
Nu, $= \dfrac{qr}{\lambda(T_w - T_\infty)}$, $q = -\lambda\left(\dfrac{\partial T}{\partial Z}\right)_w$;
Re_∞, $= r^2\alpha\omega/\nu_\infty$;
Pr, $= \mu c_p/\lambda$;
M_∞, $= \dfrac{r\omega}{a_\infty}$, $a_\infty = \sqrt{(\varkappa R T_\infty)}$.

Subscripts
∞, in motionless environment;
w, at the surface;
m, at the maximum surface radius.

REDUCTION TO THE BOUNDARY-VALUE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

CONSIDER a developed flow and heat transfer in a boundary layer at a rotating axisymmetric surface in an infinite motionless atmosphere of viscous compressible gas (Fig. 1). For calculation of a thermal boundary layer a solution of simultaneous equations for this case (p = const) is necessary (see [1]):

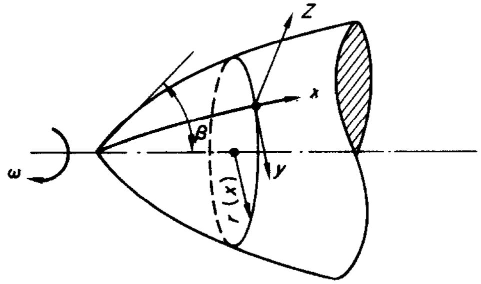

FIG. 1.

$$\left.\begin{aligned}&\frac{\partial}{\partial x}(\rho r u)+\frac{\partial}{\partial z}(\rho r w)=0\\&\rho\left(u\frac{\partial u}{\partial x}-\frac{v^2}{r}\frac{\mathrm{d}r}{\mathrm{d}x}+w\frac{\partial u}{\partial Z}\right)=\frac{\partial}{\partial Z}\left(\mu\frac{\partial u}{\partial Z}\right)\\&\rho u\frac{\partial(rv)}{\partial x}+\rho w\frac{\partial(rv)}{\partial Z}=\frac{\partial}{\partial Z}\left[\mu\frac{\partial(rv)}{\partial Z}\right]\\&\rho\left(u\frac{\partial i}{\partial x}+w\frac{\partial i}{\partial Z}\right)=\frac{1}{Pr}\frac{\partial}{\partial Z}\left(\mu\frac{\partial i}{\partial Z}\right)\\&\qquad+\mu\left[\left(\frac{\partial u}{\partial Z}\right)^2+\left(\frac{\partial v}{\partial Z}\right)^2\right].\end{aligned}\right\}\quad(1)$$

In the present case (no incoming flow) the state equation becomes

$$\rho i=\rho_\infty i_\infty. \qquad (2)$$

We shall consider the case of linear relation between viscosity and enthalpy

$$\mu=\mu_\infty\frac{i}{i_\infty}. \qquad (3)$$

Then, introduction of the stream function ψ

$$\rho r u=\rho_\infty\frac{\partial(\psi r)}{\partial Z};\quad \rho r w=-\rho_\infty\frac{\partial(\psi r)}{\partial x} \qquad (4)$$

and application of Dorodnitsyn's transformation

$$x^0=x,\quad Z^0=\int_0^z(\rho/\rho_\infty)\,\mathrm{d}Z \qquad (5)$$

yield the simultaneous equations having the form of equations for an incompressible fluid

$$\left.\begin{aligned}&\frac{\partial\psi}{\partial Z^0}\frac{\partial^2\psi}{\partial x^0\partial Z^0}-\frac{v^2}{r}\frac{\mathrm{d}r}{\mathrm{d}x^0}-\frac{1}{r}\frac{\partial(\psi r)}{\partial x^0}\frac{\partial^2\psi}{\partial Z^{02}}\\&\qquad=\nu_\infty\frac{\partial^3\psi}{\partial Z^{03}}\\&\frac{\partial\psi}{\partial Z^0}\frac{\partial(rv)}{\partial x^0}-\frac{1}{r}\frac{\partial(\psi r)}{\partial x^0}\frac{\partial(rv)}{\partial Z^0}=\nu_\infty\frac{\partial^2(rv)}{\partial Z^{02}}\\&\frac{\partial\psi}{\partial Z^0}\frac{\partial i}{\partial x^0}-\frac{1}{r}\frac{\partial(\psi r)}{\partial x^0}\frac{\partial i}{\partial Z^0}=\frac{\nu_\infty}{Pr}\frac{\partial^2 i}{\partial Z^{02}}\\&\qquad+\nu_\infty\left[\left(\frac{\partial^2\psi}{\partial Z^{02}}\right)^2+\left(\frac{\partial v}{\partial Z^0}\right)^2\right].\end{aligned}\right\}\quad(6)$$

The boundary conditions are

$$\left.\begin{aligned}&v=\omega r,\quad \psi=0,\quad \frac{\partial\psi}{\partial Z^0}=0,\\&\qquad i=i_w\quad\text{at}\quad Z^0=0\\&v\to0,\quad \frac{\partial\psi}{\partial Z^0}\to0,\quad i\to i_\infty\\&\qquad\text{at}\quad Z^0\to\infty.\end{aligned}\right\}\quad(7)$$

Consider the case of a constant temperature drop along the generating line $i_w - i_\infty$ = const.

The meridional profile of the surface (Fig. 1)

$$r=r(x) \qquad (8)$$

is assumed to have a conical nose

$$\left(\frac{\partial r}{\mathrm{d}x}\right)_{x=0}=\alpha\sin\beta \qquad (9)$$

It should be noted that the case $\alpha = 1$ corresponds to smooth blunting.

The solution of equations (6) is obtained by the method of series expansion [2] over the parameters ε_1, ε_1^2, ε_2, ε_1^3, $\varepsilon_1 . \varepsilon_2 \ldots$ where

$$\varepsilon_1 = \frac{dr}{dx} - 1, \quad \varepsilon_K = r^{K-1}\frac{d^{K-1}\varepsilon_1}{dx^{K-1}} \tag{10}$$

The solution is sought as

$$\left.\begin{aligned}
\psi &= \sqrt{(\nu_\infty \omega\alpha)}\frac{r}{\alpha}\{f_1(\zeta) + \varepsilon_1 f_2(\zeta) \\
&\quad + \varepsilon_1^2 f_3(\zeta) + \varepsilon_2 f_4(\zeta) + \varepsilon_1^3 f_5(\zeta) \\
&\quad + \varepsilon_1\varepsilon_2 f_6(\zeta) + \ldots\} \\
v &= r\omega\{g_1(\zeta) + \varepsilon_1 g_2(\zeta) + \varepsilon_1^2 g_3(\zeta) \\
&\quad + \varepsilon_2 g_4(\zeta) + \varepsilon_1^3 g_5(\zeta) + \varepsilon_1\varepsilon_2 g_6(\zeta) + \ldots\} \\
i &= i_\infty + (i_w - i_\infty)\{t_1(\zeta) + \varepsilon_1 t_2(\zeta) \\
&\quad + \varepsilon_1^2 t_3(\zeta) + \varepsilon_2 t_4(\zeta) + \varepsilon_1^3 t_5(\zeta) \\
&\quad + \varepsilon_1\varepsilon_2 t_6(\zeta) + \ldots\} + r^2\omega^2\{s_1(\zeta) \\
&\quad + \varepsilon_1 s_2(\zeta) + \varepsilon_1^2 s_3(\zeta) + \varepsilon_2 s_4(\zeta) \\
&\quad + \varepsilon_1^3 s_5(\zeta) + \varepsilon_1\varepsilon_2 s_6(\zeta) + \ldots\}
\end{aligned}\right\} \tag{11}$$

where

$f_1(\zeta), f_2(\zeta), f_3(\zeta), f_4(\zeta), f_5(\zeta), f_6(\zeta)\ldots;$
$g_1(\zeta), g_2(\zeta), g_3(\zeta), g_4(\zeta), g_5(\zeta), g_6(\zeta)\ldots;$
$t_1(\zeta), t_2(\zeta), t_3(\zeta), t_4(\zeta), t_5(\zeta), t_6(\zeta)\ldots;$

$s_1(\zeta), s_2(\zeta), s_3(\zeta), s_4(\zeta), s_5(\zeta), s_6(\zeta)\ldots$ are unknown functions of the independent variable

$$\zeta = z^0\sqrt{\left(\frac{\omega\alpha}{\nu_\infty}\right)}. \tag{12}$$

It should be noted that in this case the velocity components u and w are of the form

$$\begin{aligned}
u &= r\omega\{f_1'(\zeta) + \varepsilon_1 f_2'(\zeta) + \varepsilon_1^2 f_3'(\zeta) \\
&\quad + \varepsilon_2 f_4'(\zeta) + \varepsilon_1^3 f_5'(\zeta) + \varepsilon_1\varepsilon_2 f_6'(\zeta) + \ldots\} \\
w &= \sqrt{(\nu_\infty\omega)}\{2f_1(\zeta) + \varepsilon_1[2f_1(\zeta) + 2f_2(\zeta)] \\
&\quad + \varepsilon_1^2[2f_2(\zeta) + 2f_3(\zeta)] + \varepsilon_2[f_2(\zeta) \\
&\quad + 3f_4(\zeta)] + \varepsilon_1^3[2f_3(\zeta) + 2f_5(\zeta)] \\
&\quad + \varepsilon_1\varepsilon_2[2f_3(\zeta) + 3f_4(\zeta) + 3f_6(\zeta)] + \ldots\}
\end{aligned} \tag{13}$$

It is obvious that series (11) converge more rapidly with smaller parameters $\varepsilon_1, \varepsilon_2, \ldots$ The case $\varepsilon_K = 0$ corresponds to the conical surface. With increasing ε_K, which describe an increasing departure of $r(x)$ from the conical surface, a greater number of series terms are necessary. Substitute (11) into (6); then using the obvious identity

$$r\frac{d\varepsilon_K}{dx} = (K-1)(\varepsilon_1 + 1) + \varepsilon_{K+1}$$

and equating the factors at $\varepsilon_1^0, \varepsilon_1, \varepsilon_1^2, \varepsilon_2, \varepsilon_1^3, \varepsilon_1\cdot\varepsilon_2$ in both sides of the equations, we obtain the following recurrent systems of ordinary differential equations

$$\left.\begin{aligned}
f_1''' &= (f_1')^2 - g_1^2 - 2f_1f_1'' \\
f_2''' &= (f_1')^2 + 2f_1'f_2' - g_1^2 - 2g_1g_2 - 2f_1f_1'' - 2f_1f_2'' - 2f_1''f_2 \\
f_3''' &= 2f_2'f_2' + 2f_1'f_3' + (f_2')^2 - g_2^2 - 2g_1g_2 - 2g_1g_3 - 2f_1f_2'' - 2f_2f_1'' - 2f_1f_3'' \\
&\quad - 2f_2f_2'' - 2f_3f_1'' \\
f_4''' &= f_1'f_2' + 3f_1'f_4' - 2g_1g_4 - 3f_4f_1'' - 2f_1f_4'' - f_2f_1'' \\
f_5''' &= 2f_3'f_1' + 2f_5'f_1' + (f_2')^2 + 2f_2'f_3' - 2f_3f_1'' - 2f_2f_2'' - 2f_1f_3'' - 2f_2f_3'' - 2f_2''f_3 \\
&\quad - 2f_1f_5'' - 2f_5f_1'' - g_2^2 - 2g_1g_3 - 2g_1g_5 - 2g_2g_3 \\
f_6''' &= 2f_1'f_3' + 3f_1'f_4' + 3f_1'f_6' + (f_2')^2 + 3f_2'f_4' - 2g_1g_4 - 2g_1g_6 - 2g_2g_4 - 3f_4f_1'' \\
&\quad - 2f_1f_4'' - 2f_1f_6'' - 2f_2f_4'' - 3f_4f_2'' - 3f_6f_1'' - f_2f_2'' - 2f_3f_1''
\end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned}
g_1'' &= 2g_1f_1' - 2f_1g_1' \\
g_2'' &= 2g_1f_1' + 2f_1'g_2 + 2f_2'g_1 - 2g_2'f_1 - 2f_2g_1' - 2f_1g_1' \\
g_3'' &= 2f_1'g_3 + 2f_1'g_2 + 2f_2'g_1 + 2f_2'g_2 + 2f_3'g_1 - 2f_1g_3' - 2f_2g_2' - 2f_3g_1' - 2f_1g_2' - 2f_2g_1' \\
g_4'' &= 3f_1'g_4 + 2f_4'g_1 - 2f_1g_4' - 3f_4g_1' - f_2g_1' + f_1'g_2
\end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} g_5'' &= 2f_1'g_5 + 2f_2'g_3 + 2f_3'g_2 + 2g_1f_5' + 2f_3'g_1 + 2g_2f_2' + 2g_3f_1' - 2f_1g_5' - 2f_2g_3' \\ &\quad - 2f_3g_2' - 2f_5g_1' - 2f_1g_3' - 2f_2g_2' - 2f_3g_1' \\ g_6'' &= 3f_1'g_6 + 3f_2'g_4 + 2f_4'g_2 + 2g_1f_6' + 2g_1f_4' + 3f_1'g_4 + 2g_3f_1' + f_2'g_2 - 2f_1g_6' \\ &\quad - 2f_2g_4' - 3f_4g_2' - 3f_6g_1' - 2f_1g_4' - 3f_4g_1' - f_2g_2' - 2f_3g_1' \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{1}{Pr}t_1'' &= -2f_1t_1' \\ \frac{1}{Pr}t_2'' &= -2f_2t_1' - 2f_1t_1' - 2f_1t_2' \\ \frac{1}{Pr}t_3'' &= -2f_1t_2' - 2f_1t_3' - 2f_2t_2' - 2f_3t_1' - 2f_2t_1' \\ \frac{1}{Pr}t_4'' &= f_1't_2 + f_1't_4 - 2f_1t_4' - 3f_4t_1' - f_2t_1' \\ \frac{1}{Pr}t_5'' &= -2f_5t_1' - 2f_3t_2' - 2f_2t_3' - 2f_1t_5' - 2f_1t_3' - 2f_2t_2' - 2f_3t_1' \\ \frac{1}{Pr}t_6'' &= 2f_1't_3 + f_2't_2 + f_1't_4 + f_1't_6 + f_1't_4 - 2f_1t_6' - 2f_2t_4' - 3f_4t_2' - 3f_6t_1' - 2f_1t_4' \\ &\quad - 3f_4t_1' - t_2'f_2 - 2f_3t_1' \end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned} \frac{1}{Pr}s_1'' &= 2s_1f_1' - 2f_1s_1' - (f_1'')^2 - (g_1')^2 \\ \frac{1}{Pr}s_2'' &= 2s_1f_1' + 2s_1f_2' + 2s_2f_1' - 2f_1s_1' - 2f_1s_2' - 2f_2s_1' - 2f_1''f_2'' - 2g_1'g_2' \\ \frac{1}{Pr}s_3'' &= 2s_1f_2' + 2s_2f_1' + 2s_2f_2' + 2s_1f_3' + 2s_3f_1' - 2f_1s_2' - 2f_1s_3' - 2f_2s_1' - 2f_2s_2' \\ &\quad - 2f_3s_1' - (f_2'')^2 - 2f_1''f_3'' - 2g_1'g_3' - (g_2')^2 \\ \frac{1}{Pr}s_4'' &= 2s_1f_4' + 3s_4f_1' + s_2f_1' - 2f_1s_4' - 3f_4s_1' - f_2s_1' - 2f_1''f_4'' - 2g_1'g_4' \\ \frac{1}{Pr}s_5'' &= 2f_5's_1 + 2f_3's_2 + 2f_2's_3 + 2f_1's_5 + 2f_3's_1 + 2f_2's_2 + 2f_1's_3 - 2f_1s_5' - 2f_2s_3' \\ &\quad - 2f_3s_2' - 2f_5s_1' - 2f_1s_3' - 2f_2s_2' - 2f_3s_1' - 2f_2''f_3'' - 2f_1''f_5'' - 2g_2'g_3' - 2g_1'g_5' \\ \frac{1}{Pr}s_6'' &= 2s_1f_6' + 2s_2f_4' + 3s_4f_2' + 3s_6f_1' + 2s_1f_4' + 3f_1's_4 + s_2f_2' + 2s_3f_1' - 2f_1s_6' \\ &- 2f_2s_4' - 3f_4s_2' - 3f_6s_1' - 2f_1s_4' - 3f_4s_1' - f_2s_2' - 2f_3s_1' - 2f_2''f_4'' - 2f_1''f_6'' - 2g_2'g_4' - 2g_1'g_6'. \end{aligned}\right\} \quad (17)$$

Boundary conditions (7) are satisfied if

$$\left.\begin{aligned} &\left.\begin{aligned} &f_i = 0, \quad f_i' = 0, \quad s_i = 0, \quad (i = 1, 2, 3, \ldots) \\ &g_1 = 1, \quad g_i = 0, \quad t_1 = 1, \quad t_i = 0 \quad (i = 2, 3, \ldots) \end{aligned}\right\} \quad \text{at } \zeta = 0 \\ &g_i \to 0, \quad f_i' \to 0, \quad s_i' \to 0, \quad t_i' \to 0 \, (i = 1, 2, 3, \ldots) \qquad \text{at } \zeta \to \infty. \end{aligned}\right\} \quad (18)$$

It should be noted that the first equations of each group f_1, g_1, s_1, t_1 represent a system of non-linear differential equations describing the velocity and thermal boundary layers for cases of a disk and a cone, the solution of which is known (see [1]); the remaining equations are linear.

NUMERICAL SOLUTIONS OF BOUNDARY-VALUE PROBLEM

Boundary-value problem (14)–(18) was solved on a digital computer. The problem was reduced to the initial single-value problem by the trial and error method and interpolation of the initial conditions. Integration was carried out on the basis of the Runge–Kutta method, as modified by Merson [3], with an automatic choice of step and an accuracy $\varepsilon = 10^{-7}$ within each step. The choice of the values ζ^*, where the boundary conditions at infinity are attained, is made in the following way (see [6]). The calculations are carried out for sufficiently large ζ^* ($\zeta^* = 12$). If the results are not affected by increasing ζ^*, the solution may be considered completed. Some results are presented in Tables 1 and 2 and in Figs. 2–9. The formulae for calculation of the frictional stress components and heat-transfer coefficients at the surface of a rotating body are of the form

$$\bar{\tau}_x = Re_\infty^{-\frac{1}{2}}[f_1''(0) + \varepsilon_1 f_2''(0) + \varepsilon_1^2 f_3''(0) + \varepsilon_2 f_4''(0) + \varepsilon_1^3 f_5''(0) + \varepsilon_1\varepsilon_2 f_6''(0) + \ldots] \quad (19)$$

$$\bar{\tau}_y = Re_\infty^{-\frac{1}{2}}[g_1'(0) + \varepsilon_1 g_2'(0) + \varepsilon_1^2 g_3'(0) + \varepsilon_2 g_4'(0) + \varepsilon_1^3 g_5'(0) + \varepsilon_1\varepsilon_2 g_6'(0) + \ldots] \quad (20)$$

$$Nu = Re_\infty^{\frac{1}{2}}\{t_1'(0) + \varepsilon_1 t_2'(0) + \varepsilon_1^2 t_3'(0) + \varepsilon_2 t_4'(0) + \varepsilon_1^3 t_5'(0) + \varepsilon_1\varepsilon_2 t_6'(0) + \ldots + \theta[s_1'(0) + \varepsilon_1 s_2'(0) + \varepsilon_1^2 s_3'(0) + \varepsilon_2 s_4'(0) + \varepsilon_1^3 s_5'(0) + \varepsilon_1\varepsilon_2 s_6'(0) + \ldots]\}. \quad (21)$$

It should be noted that the temperature factor may be expressed in terms of the Mach number

Table 1. Main parameters of velocity boundary layer

Index K	$f_K''(0)$	$g_K'(0)$	$f_K(\infty)$	$-h_K(\infty)$
1	0·510233	−0·615922	0·44224	0·884472
2	0·255116	−0·307961	−0·22111	0·442245
3	−0·063779	0·076990	0·16581	−0·110601
4	0·010785	0·009424	0·04923	−0·0734097
5	0·031890	−0·038495	−0·13812	0·055396
6	−0·016177	−0·014135	−0·12307	−0·185302

Table 2. Main parameters of thermal boundary layer

$t_K'(0)$, $s_K'(0)$	Pr						
	0·1	0·3	0·7	1·0	3·0	10	30
$-t_1'(0)$	0·08284	0·18515	0·32313	0·39625	0·68258	1·13412	1·73103
$-t_2'(0)$	0·03223	0·09176	0·16154	0·19812	0·34129	0·56706	0·86552
$t_3'(0)$	0·00609	0·02104	0·04033	0·04953	0·08532	0·14176	0·21638
$t_4'(0)$	0·00461	0·01325	0·02275	0·02720	0·04371	0·06870	0·10134
$-t_5'(0)$	0·00270	0·00858	0·02007	0·02476	0·04266	0·07088	0·10819
$-t_6'(0)$	0·00588	0·01855	0·03408	0·04081	0·06557	0·10306	0·15202
$s_1'(0)$	0·04039	0·10980	0·22819	0·30796	0·75536	1·93588	4·42016
$s_2'(0)$	0·01967	0·05256	0·10974	0·14870	0·36826	0·94901	2·17341
$-s_3'(0)$	0·00514	0·01345	0·02712	0·03645	0·09127	0·24584	0·58820
$-s_4'(0)$	0·00068	0·00172	0·00347	0·00468	0·01145	0·02824	0·06090
$s_5'(0)$	0·00258	0·00678	0·01357	0·01841	0·04802	0·13641	0·34380
$s_6'(0)$	0·00107	0·00273	0·00530	0·00708	0·01809	0·05238	0·13555

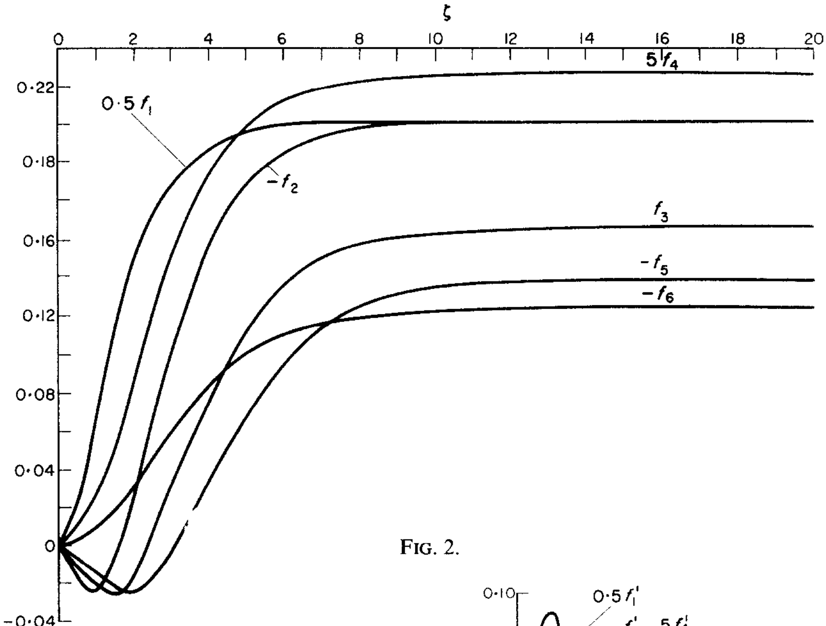

FIG. 2.

M_∞ and dimensionless temperature difference $(T_w - T_\infty)/T_\infty$

$$\theta = (\varkappa - 1)\, M_\infty^2 \frac{T_\infty}{T_w - T_\infty}.$$

The suction rate at infinity, $W(\infty)$, is also a characteristic quantity

$$- W(\infty) = \sqrt{(\nu_\infty \omega)}\{h_1(\infty) + \varepsilon_1 h_2(\infty) + \varepsilon_1^2 h_3(\infty) + \varepsilon_2 h_4(\infty) + \varepsilon_1^3 h_5(\infty) + \varepsilon_1 \varepsilon_2 h_6(\infty) + \ldots\} \qquad (22)$$

where $h_K(\infty)$ are simply expressed in terms of $f_K(\infty)$ according to (13).

SOME EXAMPLES

Consider an example of application of the results obtained to the well-studied case of a spherical surface

$$r = \sin x$$

where x and r are based on the sphere radius. In this case $\alpha = 1$,

$$\varepsilon_1 = \cos x - 1; \quad \varepsilon_1^2 = (\cos x - 1)^2;$$
$$\varepsilon_2 = -\sin^2 x; \quad \varepsilon_1^3 = (\cos x - 1)^3;$$
$$\varepsilon_1 \cdot \varepsilon_2 = -\sin^2 x\,(\cos x - 1)$$

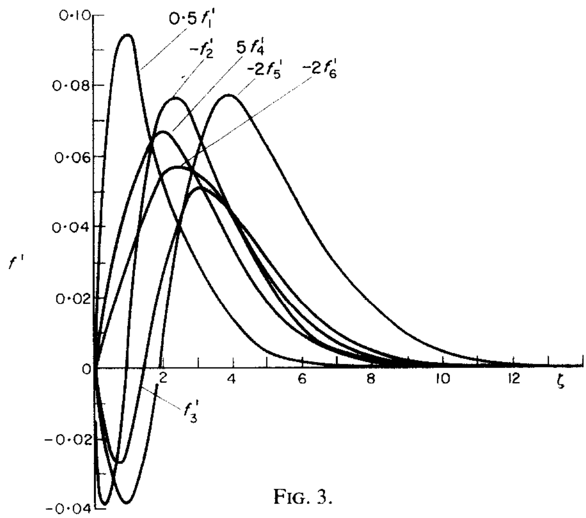

FIG. 3.

All flow parameters may be calculated directly from Tables 1 and 2. The comparison of the present calculations with the prediction of Banks [4, 5] and with the values obtained in [6] are given in Table 3. It should be noted that all the three methods may give better results with smaller x, and near the equator ($x = \pi/2$) they deviate from the true solution of the physical problem since in this region the boundary-layer equations are not valid any more. Therefore all the three methods produce similar results up to $x = 1{\cdot}0$. Table 4 shows that the

Table 3. Comparison of the results obtained by three methods (for a sphere)

Quantity	Calculation method	x: 0	0·2	0·4	0·8	1·2	1·4	1·5	$\pi/2$
$\bar{\tau}_x \cdot Re_\infty^{\frac{1}{2}}$	by equation (19)	0·5102	0·5047	0·4879	0·4180	0·2950	0·2128	0·1668	0·1325
	by Banks' method [4]	0·5102	0·5047	0·4883	0·4200	0·2947	0·2015	0·1438	0·0972
	by method [6]	0·510	0·505	0·488	0·416	0·278	0·162	(0·092)	(0·03)
$-\bar{\tau}_y \cdot Re_\infty^{\frac{1}{2}}$	by equation (20)	0·6159	0·6101	0·5927	0·5211	0·3931	0·3040	0·2528	0·2160
	by Banks' method [4]	0·6159	0·6100	0·5927	0·5213	0·3955	0·3063	0·2528	0·2106
	by method [6]	0·616	0·610	0·593	0·521	0·396	0·308	(0·259)	(0·226)
$Nu \cdot Re_\infty^{-\frac{1}{2}}$	by equation (21)	0·3231	0·3208	0·3140	0·2869	0·2372	0·1993	0·1762	0·1580
($\theta = 0$;	by Banks' method [5]	0·3231	0·3206	0·3131	0·2823	0·2283	0·1904	0·1681	0·1508
$Pr = 0{\cdot}7$)	by method [6]	0·323	0·326	0·319	0·294	0·255	0·226	(0·208)	(0·1096)
$-\frac{W(\infty)}{\sqrt{[\nu_\infty \omega]}}$	by equation (13)	0·8845	0·8786	0·8613	0·7936	0·6679	0·5696	0·5084	0·4597
	by Banks' method [4]	0·8845	0·8780	0·8587	0·7810	0·6488	0·5599	0·5091	0·4702
	by method [6]	0·884	0·879	0·864	0·806	0·709	0·641	(0·602)	(0·574)

Table 4. Values of c_m for a sphere

Obtained by the present method	Predicted by method [6]	According to Banks [4]	Experimental data [7]	Experimental data [9]
3·91	4·1	3·27	3·5	$3{\cdot}4 + \frac{22{\cdot}3}{Re^{\frac{1}{2}}}$

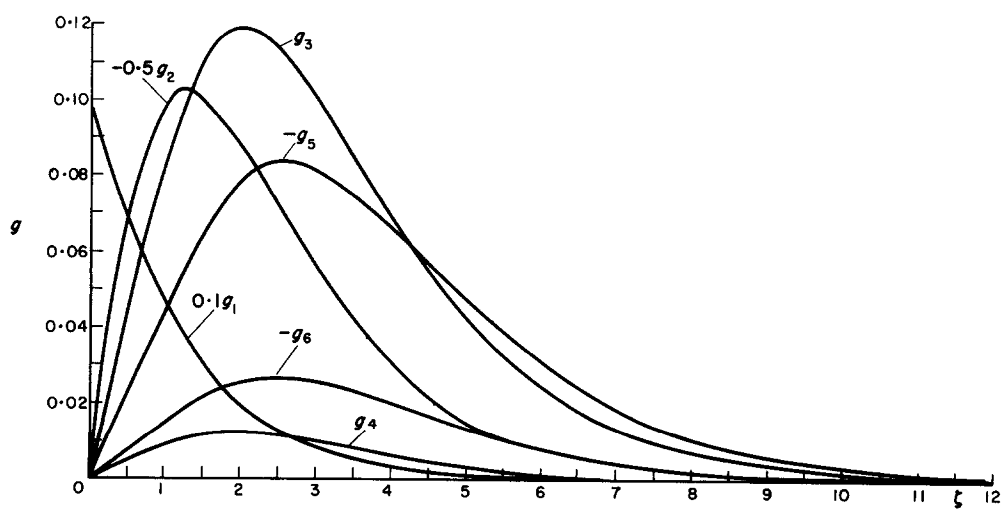

FIG. 4.

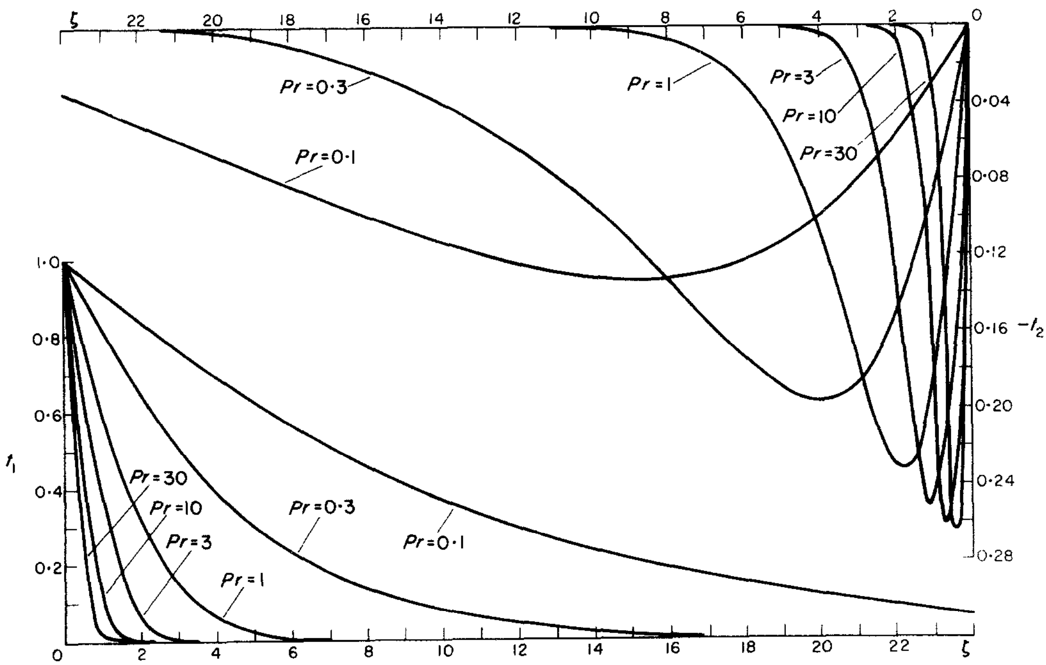
ζ
22
20
18
16
14
12
10
8
6
4
2
0
Pr = 0·3
Pr = 1
Pr = 3
Pr = 10
Pr = 30
Pr = 0·1
0·04
0·08
0·12
0·16
−f₂
0·20
0·24
0·28
1·0
0·8
0·6
f₁
0·4
0·2
Pr = 30
Pr = 10
Pr = 3
Pr = 1
Pr = 0·3
Pr = 0·1

FIG. 5.

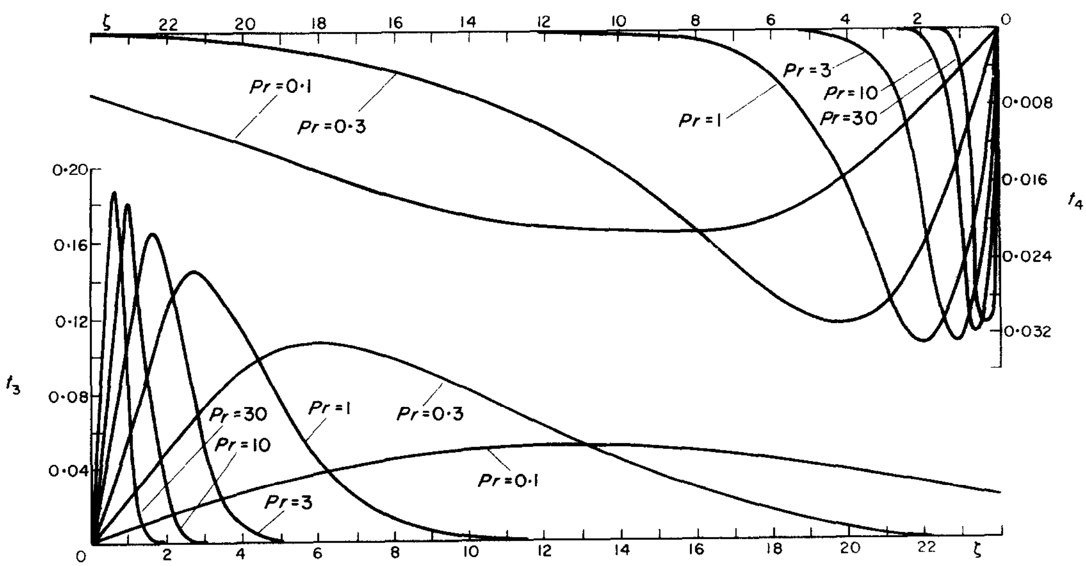
ζ
22
20
18
16
14
12
10
8
6
4
2
0
Pr = 0·1
Pr = 0·3
Pr = 1
Pr = 3
Pr = 10
Pr = 30
0·008
0·016
f₄
0·024
0·032
0·20
0·16
0·12
f₃
0·08
0·04
Pr = 30
Pr = 10
Pr = 1
Pr = 0·3
Pr = 0·1
Pr = 3

FIG. 6.

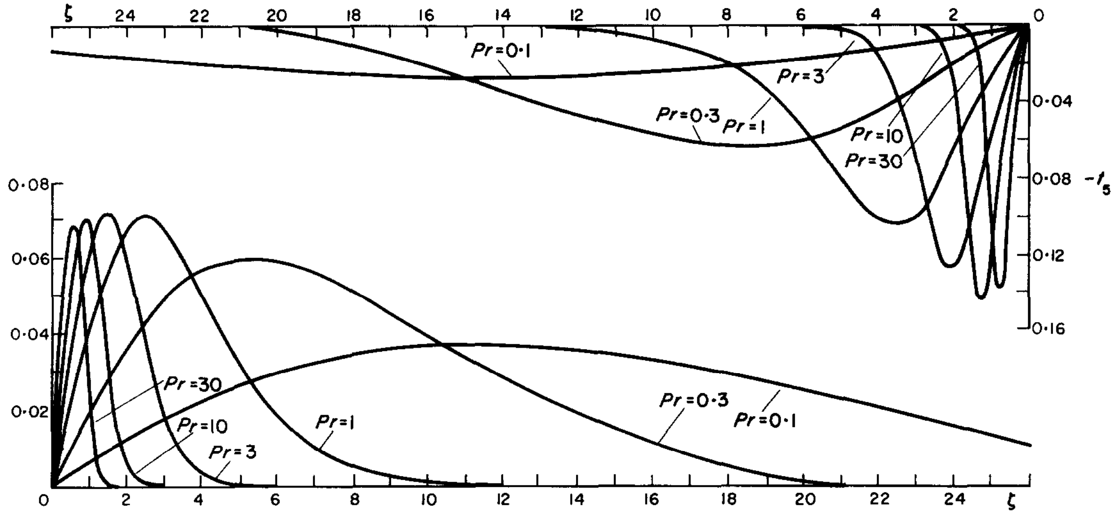
ζ
Pr=0·1
Pr=3
Pr=0·3
Pr=1
Pr=10
Pr=30
$-f_5$
Pr=30
Pr=10
Pr=3
Pr=1
Pr=0·3
Pr=0·1
ζ

FIG. 7.

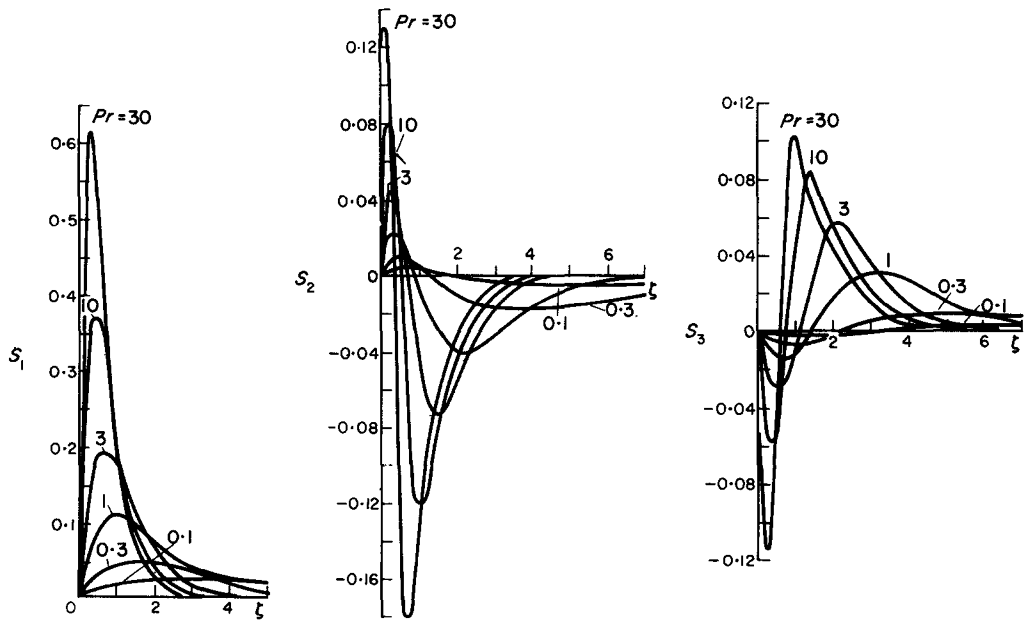
S_1
Pr=30
10
3
1
0·3
0·1
ζ
S_2
Pr=30
10
3
0·1
0·3
ζ
S_3
Pr=30
10
3
1
0·3
0·1
ζ

FIG. 8.

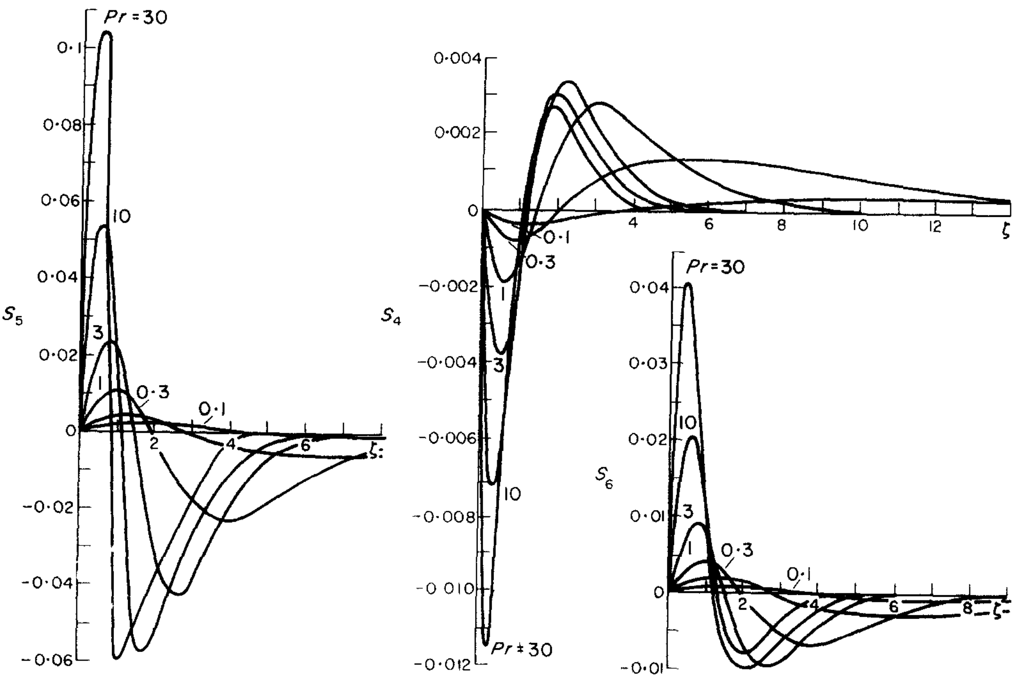
Pr = 30
10
3
1
0·3
0·1
S_5
ζ
S_4
Pr = 30
S_6
Pr = 30

FIG. 9.

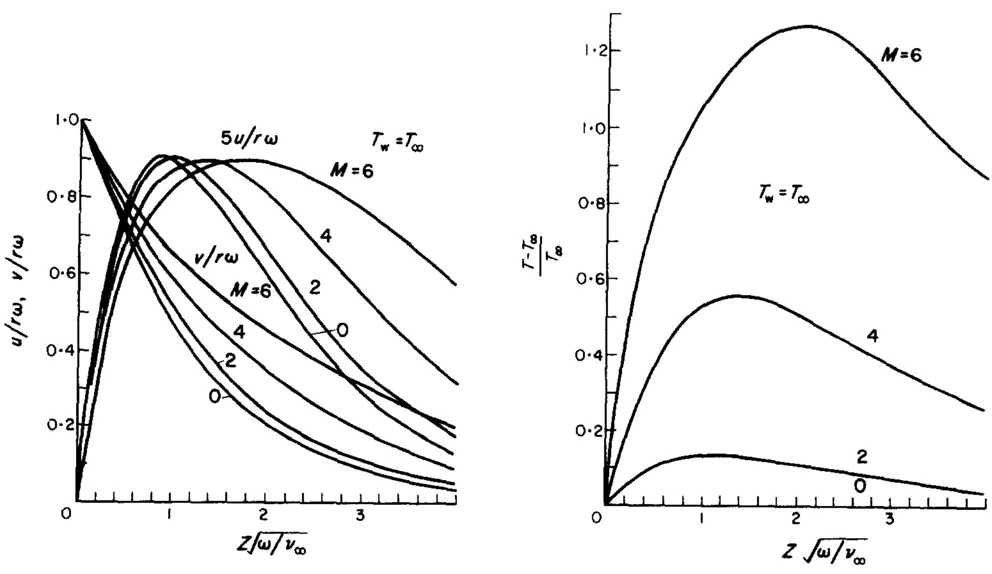
5u/rω
$T_w = T_\infty$
M = 6
4
2
0
v/rω
M = 6
u/rω, v/rω
$Z\sqrt{\omega/\nu_\infty}$
M = 6
$T_w = T_\infty$
$\frac{T-T_\infty}{T_\infty}$
$Z\sqrt{\omega/\nu_\infty}$

FIG. 10.

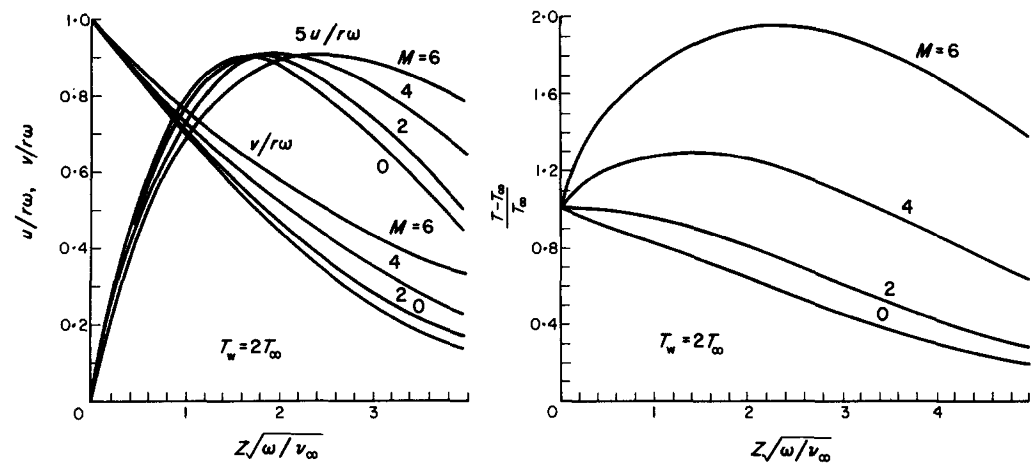

FIG. 11.

drag coefficient c_m for the sphere obtained by the present method is somewhat higher in comparison with experimental data, and Banks' values are somewhat lower. The heat-transfer coefficient obtained by the present method [6] is somewhat closer to the experimental data [8] than Banks' results. All these facts demonstrate sufficient accuracy of the present method.

For illustration of the compressibility effect on the velocity and temperature profiles, we shall take the dimensionless physical variable

$$Z\sqrt{\left(\frac{\omega\alpha}{\nu_\infty}\right)} = \int_0^\zeta \frac{i}{i_\infty}\,d\zeta$$

and for simplicity consider the case of a rotating cone (disc). The results of integration for different Mach numbers M_∞ and temperature factors T_w/T_∞ for air ($Pr = 0{\cdot}7$) show (Figs. 10 and 11) that the profiles of components of the velocity vector become less steep and the boundary layer increases with M_∞.

Heat supply to the wall has the same effect on the velocity profiles; at $T_w = 2T_\infty$ velocity profiles are less steep than in the case of heat removal ($T_w = T_\infty$). The boundary-layer thickness decreases with increasing heat removal.

It should be noted that at $\mu \sim i$ the drag coefficient c_m calculated by using physical parameters at a large distance from the rotating surface is independent of the Mach number. Experiments by Theodorsen and Regier (see [1]) verify that c_m is independent of M up to $M = 1{\cdot}69$. The same effect was reported by Bowden and Lord [7] for a sphere.

REFERENCES

1. L. A. DORFMAN, *Hydrodynamic Resistance and Heat Transfer from Rotating Bodies.* Fizmatigiz, Moscow (1960).
2. L. A. DORFMAN, On integration of the boundary layer equations for rotating axisymmetric body in infinite media, *Inzh. Zh.* **5**, 560–562 (1965).
3. G. N. LANCE, *Numerical Methods for High Speed Computers.* Iliffe, London (1960).
4. W. H. H. BANKS, The boundary layer on a rotating sphere, *Q. J. Mech. Appl. Math.* **18**, No. 4 (1965).
5. W. H. H. BANKS, Laminar boundary layer on a rotating sphere, *Z. Angew. Math. Phys.* **16**, No. 6 (1965).
6. L. A. DORFMAN and A. Z. SERAZETDINOV, Laminar flow and heat transfer near rotating axisymmetric surface, *Int. J. Heat Mass Transfer* **8**, 317–327 (1965).
7. F. G. BOWDEN and R. G. LORD, The aerodynamic resistance to a sphere rotating at high speed, *Proc. R. Soc.* **A271**, 143–153 (1963).
8. F. KREITH, L. G. ROBERTS, J. A. SULLIVAN and S. N. SINHA, Convection heat transfer and flow phenomena of rotating spheres, *Int. J. Heat Mass Transfer* **6**, 881–895 (1963).
9. O. SAWATZKI, Reibungsmoment rotierender Ellipsoide, *Strömungsmechanik und Strömungsmaschinen* No. 2 (1965).

SOLUTIONS OF EQUATIONS FOR THERMAL BOUNDARY LAYER AT ROTATING AXISYMMETRIC SURFACE

Abstract—The problem is solved by Dorodnitsyn's transformation for the case of a compressible gas, the viscosity of which is a linear function of temperature. The profiles of temperature and components of the velocity vector are presented in a series form expanded over parameters describing the shape of the meridional surface. A recurrent system of ordinary differential equations is obtained for the coefficients at the above parameters which are the functions of the dimensionless distance from the surface. The results are shown of the computer solution of the boundary-value problem for the given differential equations. The present solution for a spherical surface is compared with other solutions.

SOLUTIONS D'ÉQUATIONS POUR LA COUCHE LIMITÉE AU SURFACE ROTATOIRE À SYMÉTRIQUE

Résumé—Le problème d'un gaz compressible dans le cas où la viscosité est une fonction linéaire est résolu par la transformation de Dorodnitsyn. Les distributions de la température et des composantes du vecteur vitesse sont présentées sous la forme de séries portant sur les paramètres décrivant la forme de la surface méridienne. Un système récurrent d'équations différentielles ordinaires est obtenu pour les coefficients des paramètres ci-dessus qui sont fonctions de la distance à la surface mise sans dimensions· Les résultats de la solution obtenue par le calculateur sont présentés pour un problème de valeur à la limite pour les équations différentielles actuelles. La solution actuelle pour une surface sphérique est comparée avec d'autres solutions.

LÖSUNGEN VON THERMISCHEN GRENZSCHICHTGLEICHUNGEN FÜR ROTIERENDE ACHSSYMMETRISCHE OBERFLÄCHEN

Zusammenfassung—Das Problem des kompressiblen Gases bei linearer Temperaturabhängigkeit der Zähigkeit wird mit der Transformation von Dorodnitsyn gelöst. Die Temperaturverteilungen und Komponenten des Geschwindigkeitsvektors werden in Reihenform angegeben mit der Meridionalform als Parameter. Ein Rekursivsystem von gewöhnlichen Differentialgleichungen erhält man für die Koeffizienten bei den obigen Parametern, die Funktionen des dimensionslosen Abstands von der Oberfläche darstellen. Die Ergebnisse der Maschinenrechnung sind für ein Grenzschichtproblem angegeben. Die Lösung für eine Kugeloberfläche wird mit anderen Lösungen verglichen.